

# Parametric Ward-Takahashi identity in disordered systems and the integral identity associated with the Calogero-Sutherland model

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By utilizing the symmetric property known as the Ward-Takahashi identity in disordered systems, we explore the novel symmetry relations which hold in one-dimensional systems with inverse square interaction (the Calogero-Sutherland model). The identities emerge totally from the algebraic structure of the model. They show that the dynamical correlators are connected with one another, involving the higher-order integrals of motion. We obtain the result for the coupling strengths  $\lambda = 1/2, 1,$  and  $2,$  and conjecture that a similar relation may hold for arbitrary rational  $\lambda$ . [S1063-651X(97)11404-0]

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## I. INTRODUCTION

The Calogero-Sutherland model (CSM) describes  $N$  fermions located on a ring of the perimeter  $L$  with pairwise inverse square interactions [1,2]. The Hamiltonian is given by

$$H_{\text{CSM}} = \frac{1}{2m} \sum_i p_i^2 + \frac{\hbar^2 \lambda (\lambda - 1)}{m} \sum_{i < j} \frac{\phi^2}{\sin^2(\phi r_{ij})}, \quad (1)$$

where  $p_i = -i\hbar \partial / \partial r_i$ ,  $r_{ij} = r_i - r_j$ , and  $\phi = \pi/L$ . (The usual convention  $\hbar = 1$  and  $m = 1/2$  is adopted hereafter.) The exact ground state wave function of the model is given by a Jastrow form  $\prod_{i < j} \sin^\lambda(\phi r_{ij})$ , and excited states are known to be expressed in terms of the symmetric polynomials called the Jack polynomials [3]. While there is a long history of exactly solvable models in one-dimensional many-body systems, this model is the only family known so far where its dynamical correlation functions can be evaluated exactly. The dynamical density-density correlator  $\langle \rho(r, \tau) \rho(0, 0) \rangle$  in the thermodynamic limit was evaluated analytically for integer and rational values of  $\lambda = p/q$  [4,5]. A striking simplicity of the result emerges after taking the thermodynamic limit, which was attained through a lot of mathematical effort.

The Calogero-Sutherland model is closely connected with the random matrix theory (RMT), which has successfully been applied to describe the universal characteristics in quantum chaotic systems, such as compound nuclei, quantum billiards, and quantum dots. In the thermodynamic limit, the Jastrow form of the wave function immediately enables the ground state average to be identified with the average over Wigner-Dyson ensembles of random matrices for coupling strengths  $\lambda = \beta/2 = 1/2$  (orthogonal),  $1$  (unitary), and  $2$  (symplectic). The spectral correlator was generalized to account for spectra that disperse as a function of some external tunable parameter. Surprisingly, it was found that this parametric two-level correlator is identical to the dynamical density-density correlator of CSM [6–8]. Though this “mapping” between RMT and CSM is available only for the three special values of the coupling strengths, it can serve as a rich source of various useful insights in CSM. (See, e.g., Ref. [9] for recent work in this direction.)

In this paper, we utilize the mapping to examine the symmetric relation associated with the dynamical correlations in CSM. To do so, we extend the Ward-Takahashi identity in disordered systems to incorporate the parametric correlations with the help of the supermatrix method. In contrast to the Jack polynomial technique, this method is suitable to investigate physical quantities in the bulk limit. It also transparently provides symmetric properties to dynamical correlation functions, which we will explore.

## II. WARD-TAKAHASHI IDENTITY

Our starting point is the Ward-Takahashi identity in quantum dots or RMT. It asserts that, for the retarded and advanced Green functions  $G_E^{R,A} = (E - H \pm i0)^{-1}$ , the identity

$$\text{Tr}[\overline{G_{E_1}^R G_{E_2}^A}] = \frac{2\pi i}{\Delta(E_1 - E_2)} \quad (2)$$

is satisfied in disordered systems, where  $\Delta$  is the mean level spacing and the overbar denotes the averaging over impurity configurations or random matrices. Equation (2) results from the unitarity of the system, so it should be possible to extend this identity to incorporate the parametric dependence. Although Eq. (2) itself can be proved straightforwardly by inserting the complete diagonalized basis between  $G^R$  and  $G^A$ , such a route of derivation is no longer achieved when they carry different external parameters, since we can make no common diagonalized basis. To extend Eq. (2) to such a situation, we should take account of the unitarity of the system explicitly. To do so, we resort to the supermatrix method [10], which translates the unitarity of the system into the symmetry of the advanced and retarded components. When we present the results in the context of CSM [Eqs. (25) and (28) below], it will be found that the derived identities take quite simple forms, but still give nontrivial relations even for the free fermion case ( $\lambda = 1$ ).

## III. SUMMARY OF THE CORRELATORS IN RMT AND CSM

### A. Parametric correlators of RMT

To make our discussion concrete, take the Hamiltonian

$$H(X) = H_0 + X\Phi, \quad (3)$$

where  $H_0$  is a random matrix belonging to one of the Dyson ensembles, and  $\Phi$  is a fixed traceless member of the same ensemble. By use of the retarded and advanced Green functions

$$G_{E,X}^{R,A}(\mathbf{r},\mathbf{r}') = \langle r | [E - H(X) \pm i0]^{-1} | \mathbf{r}' \rangle, \quad (4)$$

we define the following two kinds of universal correlation functions  $k(\omega, x)$  and  $n(\omega, x)$  [11]:

$$k(\omega, x) = -\frac{1}{2} + \frac{\Delta^2}{2\pi^2} \int dr d\mathbf{r}' \overline{G_1^R(\mathbf{r}, \mathbf{r}) G_2^A(\mathbf{r}', \mathbf{r}')}, \quad (5)$$

$$n(\omega, x) = \frac{s\Delta^2}{2\pi^2} \int dr d\mathbf{r}' \overline{G_1^R(\mathbf{r}, \mathbf{r}') G_2^A(\mathbf{r}', \mathbf{r})}, \quad (6)$$

where the suffix  $i=1,2$  denotes  $(E_i, X_i)$  and  $s$  is the level degeneracy which takes account of the Kramers doublets for the symplectic case. Rescaled parameters for the energy and the external parameter were introduced to reveal the universality by

$$\omega \equiv (E_1 - E_2)/\Delta, \quad x^2 \equiv C(0)(X_1 - X_2)^2,$$

$$C(0) = \Delta^{-2} \overline{[\partial E_n(X)/\partial X]^2}.$$

For the orthogonal, unitary, and symplectic ensembles, the analytical answers for  $k(\omega, x)$  and  $n(\omega, x)$  have been obtained [11,12]. To present the results simultaneously for all three ensembles, the integral variables introduced in Ref. [5] are convenient. By assigning  $\lambda = p/q = 1/2$  for the orthogonal,  $\lambda = p/q = 1$  for the unitary, and  $\lambda = p/q = 2$  for the symplectic symmetries ( $p$  and  $q$  are coprimes), they are presented by

$$k(\omega, x) = \mathcal{I}[Q^2 e^{iQ\omega - Ex^2/2}], \quad (7)$$

$$n(\omega, x) = \mathcal{I}[E e^{iQ\omega - Ex^2/2}]. \quad (8)$$

The integration  $\mathcal{I}[\dots]$  is defined by

$$\mathcal{I}[\dots] \equiv C \prod_{i=1}^q \int_0^\infty dx_i \prod_{j=1}^p \int_0^1 dy_j F(\lambda | \{x_i, y_j\}) (\dots), \quad (9)$$

$$Q = (2\pi) \left[ \sum_{i=1}^q x_i + \sum_{j=1}^p y_j \right], \quad (10a)$$

$$E = (2\pi)^2 \left[ \sum_{i=1}^q \epsilon_p(x_i) + \sum_{j=1}^p \epsilon_H(y_j) \right], \quad (10b)$$

$\epsilon_p(x) = x(x + \lambda)$ , and  $\epsilon_H(y) = \lambda y(1 - y)$ . The numerical constant  $C$  and the form factor  $F(\lambda | \{x_i, y_j\})$  were given by

$$C = \frac{\lambda^{2p(q-1)} \Gamma^2(p) \Gamma^q(\lambda) \Gamma^p\left(\frac{1}{\lambda}\right)}{2\pi^{2p} p! q! \prod_{i=1}^q \Gamma^2(p - \lambda(i-1)) \prod_{j=1}^p \Gamma^2\left(1 - \frac{j-1}{\lambda}\right)}, \quad (11)$$

$$F(\lambda | \{x_i, y_j\}) = \frac{\prod_{i < i'} (x_i - x_{i'})^{2\lambda} \prod_{j < j'} (y_j - y_{j'})^{2\lambda}}{\prod_{i,j} (x_i + \lambda y_j)^2} \times \prod_{i=1}^q \epsilon_p(x_i)^{\lambda-1} \prod_{j=1}^p \epsilon_H(y_j)^{1/\lambda-1}. \quad (12)$$

In the supermatrix formulation,  $F(\lambda | \{x_i, y_j\})$  emerges as a Jacobian for the integration which is completely determined from the structure of the underlying graded-symmetric space.

## B. Connection with CSM

The direct connection between the parametric correlations of RMT and dynamical correlations of CSM is provided when we substitute  $\omega \rightarrow r$  and  $x^2/2 \rightarrow \tau$  ( $\tau$  is the Euclidean time) [6–8]. When we make this replacement in  $k(\omega, x)$ , it immediately reproduces the dynamical density-density correlator  $\langle \rho(r, \tau) \rho(0, 0) \rangle$  for  $\lambda = 1/2, 1$ , and  $2$ , i.e.,

$$\langle \rho(r, \tau) \rho(0, 0) \rangle = \mathcal{I}[Q^2 \cos(Qr) e^{-E\tau}]. \quad (13)$$

The other function  $n(\omega, x)$  is found to be related to the dynamical current-current correlator of CSM [9],

$$\langle j(r, \tau) j(0, 0) \rangle = \mathcal{I}[E^2 \cos(Qr) e^{-E\tau}]. \quad (14)$$

Since the Ward-Takahashi identity Eq. (2) states  $\mathcal{I}[E e^{iQ\omega}] = -1/(i\pi\omega)$ , it characterizes the current-current correlator of CSM rather than the density-density correlator.

## IV. DERIVATIONS AND RESULTS

Now we show how we can extend and derive the Ward-Takahashi identity to the case for finite  $x$ , or dynamical correlations. To avoid the notational confusion, we use  $(\omega, x)$  of RMT instead of  $(r, \tau)$  of CSM, but by substituting  $\omega \rightarrow r$  and  $x^2/2 \rightarrow \tau$ , we can obtain the corresponding expressions for CSM on each step. We follow Refs. [13,14] to derive the Ward-Takahashi identity within the framework of the supermatrix method. The basic underlying idea is to translate the unitarity of the system into the hyperbolic symmetry between the advanced and retarded components. [See Eq. (20) below.]

Consider the generating function for  $k(\omega, x)$  and  $n(\omega, x)$  in the supermatrix nonlinear- $\sigma$  model formulation [10],

$$Z_J = \langle \exp[\text{STr}(QJ)] \rangle_Q, \quad (15)$$

where  $Q$  is the  $8 \times 8$  supermatrix satisfying  $Q^2 = 1$  and its explicit structure of  $Q$  can be found in Ref. [10]. The supertrace  $\text{STr}$  is defined by  $\text{STr}(\dots) = \text{Tr}[(k_F - k_B)(\dots)]$ , where  $k_\alpha$  is a projector either onto the Bose space ( $\alpha = B$ ) or onto the Fermi space ( $\alpha = F$ ). The source matrix  $J$  is chosen as ( $a$  and  $b$  are  $c$  numbers)

$$J = (a + b \Sigma_1) \Lambda k_\alpha. \quad (16)$$

Note that we are allowed to use either one to generate  $k(\omega, x)$  and  $n(\omega, x)$ .  $\Lambda$  and  $\Sigma_1$  are  $8 \times 8$  matrices defined by

$$\Lambda = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}. \quad (17)$$

The average  $\langle \dots \rangle_{\mathcal{Q}}$  denotes the integral  $\int D\mathcal{Q}(\dots)e^{-F[\mathcal{Q}]}$ . Corresponding to  $\lambda = p/q = 1/2$  (orthogonal), 1 (unitary), and 2 (symplectic),  $F[\mathcal{Q}]$  is equal to

$$F[\mathcal{Q}] = p \left\{ \frac{i\pi\omega}{4} \text{STr}(\mathcal{Q}\Lambda) - \frac{\lambda\pi^2x^2}{16} \text{STr}(\mathcal{Q}\Lambda)^2 \right\}. \quad (18)$$

When we set the infinitesimal rotation on the saddle-point manifold as  $\mathcal{Q} \rightarrow \mathcal{Q}' = (1 - \delta T)\mathcal{Q}(1 + \delta T)$ ,  $Z_J$  remains invariant because of the integration over the manifold  $\mathcal{Q}$ . Hence

$$\begin{aligned} \delta Z_J = & \left\langle e^{\text{STr}\mathcal{Q}J} \text{STr} \left[ \left( [J, \mathcal{Q}] - \frac{pi\pi\omega}{4} [\Lambda, \mathcal{Q}] \right. \right. \right. \\ & \left. \left. \left. + \frac{p^2\pi^2x^2}{8q} \{(\Lambda\mathcal{Q})^2 - (\mathcal{Q}\Lambda)^2\} \right) \delta T \right] \right\rangle_{\mathcal{Q}} \\ = & 0. \end{aligned} \quad (19)$$

Although this identity holds for arbitrary infinitesimal rotations  $\delta T$ , we particularly choose ( $\alpha = B$  or  $F$ )

$$\delta T \propto k_{\alpha} \Sigma_1. \quad (20)$$

The choice reflects  $U(1,1)$  symmetry of the advanced and retarded components within the bosonic or fermionic sector. This hyperbolic symmetry is responsible for producing the Ward-Takahashi identity in the supermatrix method. After substituting Eq. (20) for  $\delta T$ , we have

$$\left\langle \left( b q_1 + \frac{4a - pi\pi\omega}{4} q_2 + \frac{p^2\pi^2x^2}{8q} q_3 \right) e^{a q_1 + b q_2} \right\rangle_{\mathcal{Q}} = 0, \quad (21)$$

where we define

$$q_1 = \text{STr}[k_{\alpha} \Lambda \mathcal{Q}], \quad (22a)$$

$$q_2 = \text{STr}[k_{\alpha} \Sigma_1 \Lambda \mathcal{Q}], \quad (22b)$$

$$q_3 = \text{STr}[k_{\alpha} \Sigma_1 (\Lambda \mathcal{Q})^2]. \quad (22c)$$

Note that the correlator  $k(\omega, x)$  and  $n(\omega, x)$  are related by

$$\frac{1}{16} \langle (q_1)^2 \rangle_{\mathcal{Q}} = 1 + k(\omega, x), \quad (23a)$$

$$\frac{1}{16} \langle (q_2)^2 \rangle_{\mathcal{Q}} = \pm n(\omega, x). \quad (23b)$$

Depending on the choice of  $k_B$  or  $k_F$ , we have a positive or negative sign in front of  $n(\omega, x)$ .

From Eq. (21), we can readily derive a sequence of integral identities by comparing each coefficient of polynomials of  $a$  and  $b$ . Not all of them, however, produce nontrivial integral identities. We can show that the coefficients of  $a^0 b^0$  and  $a^1 b^0$  vanish trivially. The first nontrivial identity comes from the coefficient of  $a^0 b^1$ , i.e.,

$$\frac{1}{p} \langle q_1 \rangle_{\mathcal{Q}} = \left\langle \frac{i\pi\omega}{4} (q_2)^2 - \frac{\lambda^2\pi^2x^2}{8} q_2 q_3 \right\rangle_{\mathcal{Q}}. \quad (24)$$

After some straightforward but rather lengthy evaluation of the supermatrix integration for all three values of  $\lambda$ , we obtain the result which can be summarized as follows (restoring  $\omega \rightarrow r$  and  $x^2/2 \rightarrow \tau$ ):

$$\frac{1}{\pi} = \mathcal{I}[(-irE + \tau I_3) e^{iQr - E\tau}], \quad (25)$$

where we define

$$\begin{aligned} I_n \equiv & (2\pi)^n \left[ \sum_{i=1}^q x_i (x_i + \lambda) (2x_i + \lambda)^{n-2} \right. \\ & \left. + \lambda^{n-1} \sum_{j=1}^p y_j (1 - y_j) (1 - 2y_j)^{n-2} \right]. \end{aligned} \quad (26)$$

Equation (25) serves as the extension of the Ward-Takahashi identity Eq. (2), and consists of the main result of the paper.

We can go on to the higher-order identity from Eq. (21), on principle, but the evaluation of the integration becomes harder and harder to complete. Among the second-order polynomials of  $a$  and  $b$ , we can confirm that only the coefficient of  $a^1 b^1$  gives the nontrivial relation:

$$\frac{1}{p} \langle (q_1)^2 + (q_2)^2 \rangle_{\mathcal{Q}} = \left\langle \frac{i\pi\omega}{4} q_1 (q_2)^2 - \frac{\lambda^2\pi^2x^2}{8} q_1 q_2 q_3 \right\rangle_{\mathcal{Q}}. \quad (27)$$

However, as we see from Eq. (23b), this will depend on the values of  $\lambda$  ( $p$  and  $q$ ) as well as the choice of  $\alpha = B$  or  $F$ . Hence for each value of  $\lambda$ , we have two integral identities. From these, we can seek an interesting form of the identity which seems to be the direct extension of Eq. (25), which can be presented by

$$\begin{aligned} & 1 + \mathcal{I} \left[ \left( Q^2 + \frac{q-p}{p} E \right) e^{iQr - E\tau} \right] \\ & = \frac{1}{p^2} \mathcal{I} [(-irI_3 + \tau I_4) e^{iQr - E\tau}]. \end{aligned} \quad (28)$$

Note that the coefficients of Eq. (28) are deduced to reproduce the actual results of  $\lambda = 1/2$ , 1, and 2.

## V. DISCUSSION

There are known multiple integral identities which are associated with CSM. They are called the Selberg integrals [15], and their generalization by Dotsenko and Fateev [16] are particularly useful. For instance, they were used to determine the correct normalization factor of the correlation functions [17]. We also mention that the Dotsenko-Fateev integral can provide a systematic means to evaluate a certain correlation function which showed up in disordered systems [18]. However, we emphasize that those integral formulas are not powerful enough to explain Eq. (25), because they

can be applied only when the integrands are polynomials. The simple form of the identity Eq. (25) may suggest that these known multiple integral formulas be extended somehow for the case involving an exponential factor such as  $e^{iQr-E\tau}$ . We remark that the derived integral identity Eq. (25) is not trivial at all from the mathematical point of view, even for the simplest case of the free fermion ( $\lambda = 1$ ), though we can convince ourselves of its correctness, e.g., by checking the asymptotics, or evaluating for small  $\tau$  expansion.

The quantities  $I_n$  (for  $n \geq 3$ ) correspond to the higher-order integrals of motion of CSM, as well as  $Q$  and  $I_2 = E$ . To see this transparently, identify the velocities  $v_i$  ( $\bar{v}_j$ ) for particles (holes) by

$$v_i = v_s(1 + 2x_i/\lambda), \quad (29a)$$

$$\bar{v}_j = v_s(1 - 2y_j), \quad (29b)$$

where  $v_s = \pi\lambda\hbar\rho_0/m = 2\pi\lambda$  is the sound velocity [19]. Since the velocity (rapidity) is the conserved quantity of CSM,

$$J_n = m \sum_{i=1}^q v_i^n + m_h \sum_{j=1}^p \bar{v}_j^n \quad (30)$$

should act as the integrals of motion, as does  $I_n = J_n/2 - 2\pi^2\lambda^2 J_{n-2}$ . In Ref. [20], a few kinds of the higher-order integrals of motion were investigated. Despite the similarity of their appearance, the direct connection with  $I_n$  in Eq. (26) is missing at present.

## VI. CONCLUSION

In conclusion, we have derived the Ward-Takahashi identity for the parametric correlations of RMT. By doing so, it was shown that there exist novel integral identities which are associated with the dynamical correlations of CSM. It is remarked that they amount to a new generalization of the Selberg integration. As was seen from the derivation in the context of RMT, this is the manifestation of the unitarity of the system, i.e., the hyperbolic symmetry of the advanced and retarded components. However, its nature and implication in CSM is not so clear at present. Since our arguments rely heavily on the mapping between RMT and CSM, we can make no decisive statement on the validity of the derived integral identities for *arbitrary rational* values of  $\lambda$ . We can, however, suggest two possible scenarios: Eqs. (25) and (28) are (i) true only for  $\lambda = 1/2, 1, 2$ , or (ii) true for all rational values of  $\lambda$ . If the latter were true, it would remain as a future challenge how the integral identities Eqs. (25) and (28) can be deduced from the Jack polynomials, or the  $W$  algebra, which is known as the symmetry of CSM [21].

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- [1] F. Calogero, J. Math. Phys. **10**, 2191 (1969).  
 [2] B. Sutherland, J. Math. Phys. **12**, 246 (1971).  
 [3] P. J. Forrester, Nucl. Phys. **B416**, 377 (1994).  
 [4] F. Lesage, V. Pasquier, and D. Serban, Nucl. Phys. **B435[FS]**, 585 (1995).  
 [5] Z. N. C. Ha, Nucl. Phys. **B435[FS]**, 604 (1995).  
 [6] B. D. Simons, P. A. Lee, and B. L. Altshuler, Nucl. Phys. **B409[FS]**, 487 (1993).  
 [7] C. W. J. Beenakker, Phys. Rev. Lett. **70**, 4126 (1993).  
 [8] O. Narayan and B. S. Shastry, Phys. Rev. Lett. **71**, 2106 (1993).  
 [9] N. Taniguchi, B. S. Shastry, and B. L. Altshuler, Phys. Rev. Lett. **75**, 3724 (1995).  
 [10] K. B. Efetov, Adv. Phys. **32**, 53 (1983).  
 [11] B. D. Simons and B. L. Altshuler, Phys. Rev. B **48**, 5422 (1993).  
 [12] N. Taniguchi, A. V. Andreev, and B. L. Altshuler, Europhys. Lett. **29**, 515 (1995).  
 [13] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. **129**, 367 (1985).  
 [14] J. A. Zuk (unpublished).  
 [15] M. L. Mehta, *Random Matrices — Revised and Enlarged Second Edition* (Academic, San Diego, 1991).  
 [16] V. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B251[FS13]**, 691 (1985).  
 [17] P. J. Forrester, Mod. Phys. Lett. **B9**, 359 (1995).  
 [18] P. J. Forrester and J. A. Zuk, Nucl. Phys. **B473**, 616 (1996).  
 [19] M. R. Zirnbauer and F. D. M. Haldane, Phys. Rev. B **52**, 8729 (1995).  
 [20] R. A. Römer, B. S. Shastry, and B. Sutherland, J. Phys. A **29**, 4699 (1996).  
 [21] K. Hikami and M. Wadati, J. Phys. Soc. Jpn. **62**, 4203 (1993).